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ON A PURSUIT GAME ON CAYLEY GRAPHS

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The game cops and robbers is considered on Cayley graphs of abelian groups. It is proved that if the graph has degree d, then $\lceil (d+1)/2 \rceil$ cops are sufficient to catch one robber. This bound is often best possible.

1. Introduction

We consider the following game, called cops and robbers. There is a finite, connected, undirected graph $G=(V, E)$, m cops and one robber. First the cops choose one vertex each as initial position. Next the robber makes his choice. Afterwards they move alternately along the edges of the graph or stay. (First the cops, then the robber.) The game is with full information, that is, everyone knows the position of the others.

Denote by $c(G)$ the minimum value of m for which m cops have a winning strategy, i.e., they have an algorithm to catch the robber (at least one cop gets to the same vertex as the robber) no matter how he plays.

This game was studied by several authors: Aigner and Fromme [1], Andreae [2], [3], Hamidoune [5], Quilliot [9], [10], Maamoun and Meyniel [7].

For a finite group H and a subset S of its elements satisfying $S=S^{-1}$ one defines the Cayley graph $C(H, S)=(H, E)$ with vertex set H and edge set $E=\{\{h; hs\}: h\in H, s\in S\}$. Let $H_0=\langle S\rangle$ be the subgroup of H generated by S. Then $C(H, S)$ is connected if and only if $H=H_0$. Otherwise it is the disjoint union of $|H:H_0|$ copies of the connected Cayley graph $C(H_0, S)$.

The main result of this paper is the following:

Theorem 1. *Suppose that $C(H, S)$ is a connected Cayley graph of the abelian group H. Then we have*

$$c\big(C(H, S)\big) \leqq \lceil (|S|+1)/2 \rceil. \tag{1.1}$$

Inequality (1.1) improves the bound $\lceil 3|S|/4 \rceil$, which was obtained by Hamidoune [5], whose ideas are used in the proof of (1.1).

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Let us note also that the assumption on the commutativity of H cannot be dropped. Using the constructions of Margulis [8] and Imrich [6] for Cayley graphs with large girth, it is shown in [4] that there exist Cayley graphs G of degree d, for every $d \geqq 3$, and with $c(G)$ arbitrarily large. Let C_n denote the cycle of length n and let $\delta(G)$ denote the minimum degree of G.

Aigner and Fromme [1] showed that if G does not contain C_3 and C_4 then $c(G) \geqq \delta(G)$ holds.

Their proof can be adapted to show the following:

Proposition 2. *Suppose that in G any two vertices are connected by at most 2 paths of length at most 2. Then $c(G) \geqq \delta(G)/2$. Moreover, if G contains no C_3 then $c(G) \geqq (\delta(G)+1)/2$ holds.*

This proposition can be used to construct many Cayley graphs G with $c(G) \geqq (\delta(G)+1)/2$, i.e., giving equality in Theorem 1. In particular, if S is a minimal generating set of the abelian group H, then Theorem 1 and Proposition 2 imply $c(H, S) = \lceil (|S|+1)/2 \rceil$.

2. The proof of Theorem 1

To prove the theorem let us introduce the following restricted version of the game cops and robbers on $C(H, S)$. Let T be an arbitrary subset of S and suppose that from vertex h the robber can move only to one of the vertices $\{ht : t \in T\}$, or stay in h.

That is, the robber can only use the generators in T while the cops those of S.

Let $c(H, S, T)$ denote the minimum number of cops needed to catch the robber in this restricted game, where $H = \langle s \rangle$ is assumed.

Theorem 3. *For every finite abelian group H and all subsets $T \subseteqq S \subseteqq H$ one has $c(H, S, T) \leqq \lceil (|T|+1)/2 \rceil$.*

Note that Theorem 1 is the special case $S = T$.

Proof of Theorem 3. We prove Theorem 3 by double induction: we apply induction on $|H|$ and for fixed H on $|T|$.

The case $|T| \leqq 1$ is trivial: the one and only cop moves in a finite number of steps to the connected component of $C(H, T)$ where the robber is. Note that the robber cannot leave this component. If $T = \emptyset$, then the robber is caught. If $T = \{t\}$, then the cop keeps using t^{-1} and catches the robber in less than the order of t steps.

Also, if $|T| = 2$, $T = \{t, t^{-1}\}$, then $c(H, S, T) \leqq \lceil (2+1)/2 \rceil = 2$ follows easily. Both cops move to the connected component of $C(H, T)$ containing the robber. Then one of them keeps using t, the other t^{-1} and they catch the robber in less than half of the order of t moves.

From now on we may assume that there are two elements $s, t \in T$ such that $st \neq 1$.

Let $K = \langle st \rangle$ be the group generated by st.

Define $\bar{H}=H/K$, $\bar{T}=TK/K$, $\bar{S}=SK/K$. Then $|\bar{H}|<|H|$ and $|\bar{T}|\leqq|T|$ imply, by the induction hypothesis

$$c(\bar{H}, \bar{S}, \bar{T}) \leqq \left\lceil \frac{|T|+1}{2} \right\rceil \stackrel{\text{def}}{=} b.$$

That is, b cops have a winning strategy in the restricted game with parameters $\bar{H}$, $\bar{S}$ and $\bar{T}$.

Suppose that there are b cops and one robber in the game with parameters H, S and T. Then the cops evaluate their position and the position of the robber modulo K (i.e., via the homomorphism $h\to hK/K$) and apply their winning strategy in $\bar{H}$. After a finite number of steps one cop, say the first one, will catch the robber in $\bar{H}$, that is, if the first cop is in vertex h_1 and the robber in h_2 then $h_1^{-1}h_2=(st)^j$ holds for some non negative integer j. If $j=0$, then the game is finished. So suppose $j\geqq 1$ and we describe the strategy of the cops from this moment on.

If the robber uses $s(t)$ then the first cop uses $t^{-1}(s^{-1})$, respectively. If the robber uses some $r\in T-\{t, s\}$ or if he stays then the first cop does the same. Now it is clear that once the robber used s and t altogether j times he will be caught by the first cop.

To conclude the proof, we must show that the remaining $b-1$ cops can always force the robber to use s or t. In fact, they will play, as if it was the game with parameters $H, S, T-\{s, t\}$, where they have a winning strategy by induction. Thus if the robber would never use s or t, i.e., if he also played the same game, then he would be caught. Therefore, the robber will have to use s or t, concluding the proof. ∎

Remark. Further analysis of the proof shows that the robber can be caught in at most $|H|\left\lceil \frac{|T|+1}{2} \right\rceil$ moves.

3. Proof of Proposition 2

Set $b=c(G)$. Since the graph is connected, if the cops have a winning strategy, then they can win from an arbitrary initial position. So we can suppose that the robber is not caught in the first move. As he is supposed to lose, there must be a final position, where he cannot escape, i.e., a vertex v and b other vertices $u_1, \dots, u_b$ so that v and every neighbor of v are connected to some u_i, $1\leqq i\leqq b$.

By assumption, if u_i is not a neighbor of v then they can have at most two common neighbors. If u_i is a neighbor of v then they can have at most one common neighbor, plus u_i itself, plus v, which should be counted only at one neighbor. Thus we have $2b+1\geqq \deg(v)+1$, or $c(G)\geqq\delta(G)/2$, giving the first part of the proposition.

To prove the second, note that if u_i is a neighbor of v and G contains no C_3, then u_i and v have no common neighbor. This gives $2b\geqq\deg(v)+1$, or $c(G)\geqq$ $\geqq(\delta(G)+1)/2$. ∎

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